

# A note on Mahler's conjecture

Shlomo Reisner, Carsten Schütt and Elisabeth M. Werner <sup>\*†</sup>

## Abstract

Let  $K$  be a convex body in  $\mathbb{R}^n$  with Santaló point at 0. We show that if  $K$  has a point on the boundary with positive generalized Gauß curvature, then the volume product  $|K||K^\circ|$  is not minimal. This means that a body with minimal volume product has Gauß curvature equal to 0 almost everywhere and thus suggests strongly that a minimal body is a polytope.

## 1 Introduction

A convex body  $K$  in  $\mathbb{R}^n$  is a compact, convex set with nonempty interior. The polar body  $K^z$  with respect to an interior point  $z$  of  $K$  is

$$K^z = \{y \mid \forall x \in K : \langle y, x - z \rangle \leq 1\}.$$

There is a unique point  $z \in K$  such that the volume product  $|K||K^\circ|$  is minimal. This point is called the Santaló point  $s(K)$ . The Blaschke-Santaló inequality asserts that the maximum of the volume product  $|K||K^\circ|$  is attained for all ellipsoids and only for ellipsoids [2, 16, 11]. Thus the convex body for which the maximum is attained is unique up to affine transforms.

On the other hand, it is an open problem for which convex bodies the minimum is attained. It is conjectured that the minimum is attained for the simplex. The class of centrally symmetric convex bodies is of particular importance. Mahler conjectured [8, 9] that the minimum in this class is attained for the cube and its polar body, the cross-polytope. If so, the minimum would also be attained by “mixtures” of the cube and the cross-polytope, sometimes called Hanner-Lima bodies. Those are not affine images of the cube or the cross-polytope. Thus, in the class of centrally symmetric convex bodies the minimum is not attained for a unique convex body (up to affine transforms).

The first breakthrough towards Mahler's conjecture is the inequality of Bourgain-Milman [3]. They proved that for centrally symmetric convex bodies

$$\left(\frac{c}{n}\right)^n \leq |K||K^\circ|.$$

---

Keywords: convex bodies, volume product, Mahler's conjecture.

2000 Mathematics Subject Classification: 52A20.

<sup>\*</sup>Part of the work was done while the three authors visited the Renyi Institute of Mathematics of the Hungarian Academy of Sciences, in the summer of 2008 and during a workshop at the American Institute of Mathematics in Palo Alto in August of 2010. They are indebted to both Institutes.

<sup>†</sup>E. Werner has been partially supported by an NSF grant, a FRG-NSF grant and a BSF grant.

This inequality has recently been reproved with completely different methods by Kuperberg and Nazarov [7, 12]. Their proofs also give better constants.

For special classes like zonoids and unconditional bodies Mahler's conjecture has been verified [13, 5, 15, 10, 14].

The inequality of Bourgain-Milman has many applications in various fields of mathematics: geometry of numbers, Banach space theory, convex geometry, theoretical computer science.

Despite great efforts, a proof of Mahler's conjecture seems still elusive. It is not even known whether a convex body for which the minimum is attained must be a polytope. A result in this direction has been proved by Stancu [20]. It is shown there that if  $K$  is of class  $C^2$  with strictly positive Gauß curvature everywhere, then the volume product of  $K$  can not be a local minimum.

In this paper we show that a minimal body can not have even a single point with positive generalized curvature. By a result of Alexandrov, Busemann and Feller [1, 4] the generalized curvature exists almost everywhere. Therefore, our result implies that a minimal body has almost everywhere curvature equal to 0 and thus suggests strongly that a minimal body is a polytope.

We now introduce the concept of generalized curvature. Let  $\mathcal{U}$  be a convex, open subset of  $\mathbb{R}^n$  and let  $f : \mathcal{U} \rightarrow \mathbb{R}$  be a convex function.  $df(x) \in \mathbb{R}^n$  is called subdifferential at the point  $x_0 \in \mathcal{U}$ , if we have for all  $x \in \mathcal{U}$

$$f(x_0) + \langle df(x_0), x - x_0 \rangle \leq f(x).$$

A convex function has a subdifferential at every point and it is differentiable at a point if and only if the subdifferential is unique. Let  $\mathcal{U}$  be an open, convex subset in  $\mathbb{R}^n$  and  $f : \mathcal{U} \rightarrow \mathbb{R}$  a convex function.  $f$  is said to be twice differentiable in a generalized sense in  $x_0 \in \mathcal{U}$ , if there is a linear map  $d^2f(x_0) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and a neighborhood  $\mathcal{U}(x_0) \subseteq \mathcal{U}$  such that we have for all  $x \in \mathcal{U}(x_0)$  and for all subdifferentials  $df(x)$

$$\|df(x) - df(x_0) - d^2f(x_0)(x - x_0)\| \leq \Theta(\|x - x_0\|)\|x - x_0\|.$$

Here,  $\| \cdot \|$  is the standard Euclidean norm on  $\mathbb{R}^n$  and  $\Theta$  is a monotone function with  $\lim_{t \rightarrow 0} \Theta(t) = 0$ .  $d^2f(x_0)$  is called (generalized) Hesse-matrix. If  $f(0) = 0$  and  $df(0) = 0$  then we call the set

$$\{x \in \mathbb{R}^n | x^t d^2f(0)x = 1\}$$

the indicatrix of Dupin at 0. Since  $f$  is convex, this set is an ellipsoid or a cylinder with a base that is an ellipsoid of lower dimension. The eigenvalues of  $d^2f(0)$  are called generalized principal curvatures and their product is called the generalized Gauß-Kronecker curvature  $\kappa$ .

It will always be this generalized Gauß curvature that we mean throughout the rest of the paper though we may occasionally just call it Gauß curvature. Geometrically the eigenvalues of  $d^2f(0)$  that are different from 0 are the lengths of the principal axes of the indicatrix raised to the power  $(-2)$ .

To define the generalized Gauß curvature  $\kappa(x)$  of a convex body  $K$  at a boundary point  $x$  with unique outer normal  $N_K(x)$ , if it exists, we translate and rotate  $K$  so that we may

assume that  $x = 0$  and  $N_K(x) = -e_n$ .  $\kappa(x)$  is then defined as the Gauß curvature of the function  $f : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$  whose graph in the neighborhood of 0 is  $\partial K$ .

We further denote by  $H(x, \xi)$  the hyperplane through  $x$  and orthogonal to  $\xi$ .  $H^-(x, \xi)$  and  $H^+(x, \xi)$  are the two half spaces determined by  $H(x, \xi)$ . In particular, for  $\Delta > 0$ , a convex body  $K$  and  $x \in \partial K$ , the boundary of  $K$ , with a unique outer normal  $N_K(x)$

$$H(x - \Delta N_K(x), N_K(x))$$

is the hyperplane through  $x - \Delta N_K(x)$  with normal  $N_K(x)$ .  $H^+(x - \Delta N_K(x), N_K(x))$  denotes the halfspace determined by  $H(x - \Delta N_K(x), N_K(x))$  that does not contain  $x$ .

We construct two new bodies,  $K_x(\Delta)$ , by cutting off a cap

$$K_x(\Delta) = K \cap H^+(x - \Delta N_K(x), N_K(x)),$$

and  $K^x(\Delta)$  by

$$K^x(\Delta) = \text{co}[K, x + \Delta N_K(x)].$$

## 2 The main theorem

**Theorem 1.** *Let  $K$  be a convex body in  $\mathbb{R}^n$  and suppose that there is a point in the boundary of  $K$  where the generalized Gauß curvature exists and is not 0. Then the volume product  $|K||K^{s(K)}|$  is not a local minimum.*

*Moreover, if  $K$  is centrally symmetric with center 0 then, under the above assumption, the volume product  $|K||K^\circ|$  is not a local minimum in the class of 0-symmetric convex bodies.*

In order to prove Theorem 1, we present the following proposition.

**Proposition 2.** *Let  $K$  be a convex body in  $\mathbb{R}^n$  whose Santaló point is at the origin. Suppose that there is a point  $x$  in the boundary of  $K$  where the generalized Gauß curvature exists and is not 0. Then there exists  $\Delta > 0$  such that*

$$|K_x(\Delta)||K_x(\Delta)^\circ| < |K||K^\circ|.$$

or

$$|K^x(\Delta)||K^x(\Delta)^\circ| < |K||K^\circ|.$$

For the proof of Proposition 2 we need several lemmas from [18] and [19]. We refer to [18] and [19] for the proofs. In particular, part (ii) of this lemma can be found in [19] as Lemma 12.

**Lemma 3.** [19] *Let  $K$  be a convex body in  $\mathbb{R}^n$ . Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear, invertible map.*  
*(i) The normal at  $T(x)$  is*

$$(T^{-1})^t(N_K(x))\|(T^{-1})^t(N_K(x))\|^{-1}.$$

*(ii) Suppose that the generalized Gauß-Kronecker curvature  $\kappa$  exists in  $x \in \partial K$ . Then the generalized Gauß-Kronecker curvature  $\kappa$  exists in  $T(x) \in \partial T(K)$  and*

$$\kappa(x) = \|(T^{-1})^t(N_K(x))\|^{n+1} \det(T)^2 \kappa(T(x)).$$

The next two lemmas are well known. See e.g. [18].

**Lemma 4.** [18] *Let  $\mathcal{U}$  be an open, convex subset of  $\mathbb{R}^n$  and  $0 \in \mathcal{U}$ . Suppose that  $f : \mathcal{U} \rightarrow \mathbb{R}$  is twice differentiable in the generalized sense at 0 and that  $f(0) = 0$  and  $df(0) = 0$ . Suppose that the indicatrix of Dupin at 0 is an ellipsoid. Then there is a monotone, increasing function  $\psi : [0, 1] \rightarrow [1, \infty)$  with  $\lim_{s \rightarrow 0} \psi(s) = 1$  such that*

$$\begin{aligned} & \left\{ (x, s) \left| x^t d^2 f(0) x \leq \frac{2s}{\psi(s)} \right. \right\} \\ & \subseteq \{(x, s) | f(x) \leq s\} \subseteq \{(x, s) | x^t d^2 f(0) x \leq 2s\psi(s)\}. \end{aligned}$$

**Lemma 5.** [18] *Let  $K$  be a convex body in  $\mathbb{R}^n$  with  $0 \in \partial K$  and  $N(0) = -e_n$ . Suppose that the indicatrix of Dupin at 0 is an ellipsoid. Suppose that the principal axes  $b_i e_i$  of the indicatrix are multiples of the unit vectors  $e_i$ ,  $i = 1, \dots, n-1$ . Let  $\mathcal{E}$  be the  $n$ -dimensional ellipsoid*

$$\mathcal{E} = \left\{ x \in \mathbb{R}^n \left| \sum_{i=1}^{n-1} \frac{x_i^2}{b_i^2} + \frac{\left(x_n - \left(\prod_{i=1}^{n-1} b_i\right)^{\frac{2}{n-1}}\right)^2}{\left(\prod_{i=1}^{n-1} b_i\right)^{\frac{2}{n-1}}} \leq \left(\prod_{i=1}^{n-1} b_i\right)^{\frac{2}{n-1}} \right. \right\}.$$

*Then there is an increasing, continuous function  $\phi : [0, \infty) \rightarrow [1, \infty)$  with  $\phi(0) = 1$  such that we have for all  $t$*

$$\begin{aligned} & \left\{ \left( \frac{x_1}{\phi(t)}, \dots, \frac{x_{n-1}}{\phi(t)}, t \right) \left| x \in \mathcal{E}, x_n = t \right. \right\} \\ & \subseteq K \cap H((0, \dots, 0, t), N(0)) \\ & \subseteq \{(\phi(t)x_1, \dots, \phi(t)x_{n-1}, t) | x \in \mathcal{E}, x_n = t\}. \end{aligned}$$

*We call  $\mathcal{E}$  the standard approximating ellipsoid.*

Let us denote the lengths of the principal axes of the indicatrix of Dupin by  $b_i$ ,  $i = 1, \dots, n-1$ . Then the lengths  $a_i$ ,  $i = 1, \dots, n$  of the principal axes of the standard approximating ellipsoid  $\mathcal{E}$  are

$$a_i = b_i \left( \prod_{i=1}^{n-1} b_i \right)^{\frac{1}{n-1}} \quad i = 1, \dots, n-1 \quad \text{and} \quad a_n = \left( \prod_{i=1}^{n-1} b_i \right)^{\frac{2}{n-1}}. \quad (1)$$

This follows immediately from Lemma 5. For the generalized Gauß-Kronecker curvature we get

$$\prod_{i=1}^{n-1} \frac{a_n}{a_i^2}. \quad (2)$$

This follows as the generalized Gauß-Kronecker curvature equals the product of the eigenvalues of the generalized Hesse matrix. The eigenvalues are  $b_i^{-2}$ ,  $i = 1, \dots, n-1$ . Thus

$$\prod_{i=1}^{n-1} b_i^{-2} = \left( \prod_{i=1}^{n-1} b_i \right)^{2} \prod_{i=1}^{n-1} \left( b_i \left( \prod_{k=1}^{n-1} b_k \right)^{\frac{1}{n-1}} \right)^{-2} = \prod_{i=1}^{n-1} \frac{a_n}{a_i^2}.$$

In particular, if the indicatrix of Dupin is a sphere of radius  $\sqrt{\rho}$  then the standard approximating ellipsoid is a Euclidean ball of radius  $\rho$ .

We consider the map  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$

$$T(x) = \left( \frac{x_1}{a_1} \left( \prod_{i=1}^{n-1} b_i \right)^{\frac{2}{n-1}}, \dots, \frac{x_{n-1}}{a_{n-1}} \left( \prod_{i=1}^{n-1} b_i \right)^{\frac{2}{n-1}}, x_n \right). \quad (3)$$

This transforms the standard approximating ellipsoid  $\mathcal{E}$  into a Euclidean ball  $T(\mathcal{E})$  with radius  $r = (\prod_{i=1}^{n-1} b_i)^{2/(n-1)}$ . This is obvious since the principal axes of the standard approximating ellipsoid are given by (1). The map  $T$  is volume preserving.

**Lemma 6.** *Let  $K$  be a convex body in  $\mathbb{R}^n$  with  $0$  as an interior point. Suppose that the generalized Gauß curvature of  $\partial K$  at  $x_0$  exists and that the indicatrix is an ellipsoid. Then there is an invertible linear transformation  $T$  such that*

$$(i) \ N_{T(K)}(T(x_0)) = \frac{T(x_0)}{\|T(x_0)\|}$$

(ii) *The indicatrix of Dupin at  $T(x_0)$  is a Euclidean ball.*

$$(iii) \ |T(K)| = |T(K)^\circ|$$

$$(iv) \ \|T(x_0)\| = 1.$$

**Proof.** (i) We first show that there is a linear map  $T_1$  such that  $N_{T_1(K)}(T_1(x_0)) = \frac{T_1(x_0)}{\|T_1(x_0)\|}$ . Let  $e_i$ ,  $1 \leq i \leq n$  be an orthonormal basis of  $\mathbb{R}^n$  such that  $e_n = N_K(x_0)$ . Define  $T_1$  by

$$T_1 \left( \sum_{i=1}^{n-1} t_i e_i + t_n x_0 \right) = \sum_{i=1}^{n-1} t_i e_i + t_n \langle x_0, e_n \rangle e_n$$

$T_1$  is well-defined since  $x_0$  and  $e_1, \dots, e_{n-1}$  are linearly independent. Indeed, as  $\langle e_n, x_0 \rangle > 0$ ,  $x_0 \notin e^\perp = \text{span}\{e_1, \dots, e_{n-1}\}$ . Moreover,

$$T_1(x_0) = \langle x_0, e_n \rangle e_n$$

and thus

$$\frac{T_1(x_0)}{\|T_1(x_0)\|} = e_n \quad (4)$$

By Lemma 3, the outer normal at  $T_1(x_0)$  is

$$(T_1^{-1})^t(N_K(x_0)) \|(T_1^{-1})^t(N_K(x_0))\|^{-1}.$$

Then for all  $1 \leq i \leq n-1$

$$\langle (T_1^{-1})^t(N_K(x_0)), e_i \rangle = \langle N_K(x_0), T_1^{-1} e_i \rangle = \langle N_K(x_0), e_i \rangle = \langle e_n, e_i \rangle = 0.$$

Hence

$$\frac{(T_1^{-1})^t(N_K(x_0))}{\|(T_1^{-1})^t(N_K(x_0))\|} = \pm e_n$$

Since  $\langle x_0, e_n \rangle > 0$  and since 0 is an interior point, it is  $+e_n$ . Together with (4), this shows that

$$N_{T_1(K)}(T_1(x_0)) = e_n = \frac{T_1(x_0)}{\|T_1(x_0)\|}.$$

(ii) Put  $x_1 = T_1(x_0)$  and  $K_1 = T_1(K)$ . By Lemma 3, the curvature  $\kappa(x_1)$  at  $x_1 \in \partial K_1$  exists and is positive. For  $1 \leq i \leq n-1$ , let  $b_i$  be the principal curvatures and  $a_i$  be the principal axes of the standard approximating ellipsoid in  $x_1$ . Let  $T_2 : \mathbb{R}^n \rightarrow \mathbb{R}^n$

$$T_2(x) = \left( \frac{\xi_1}{a_1} \left( \prod_{i=1}^{n-1} b_i \right)^{\frac{2}{n-1}}, \dots, \frac{\xi_{n-1}}{a_{n-1}} \left( \prod_{i=1}^{n-1} b_i \right)^{\frac{2}{n-1}}, \xi_n \right). \quad (5)$$

transforms the indicatrix of Dupin at  $x_1$  into an  $n-1$ -dimensional Euclidean ball and the standard approximating ellipsoid  $\mathcal{E}$  into a  $n$ -dimensional Euclidean ball  $T_2(\mathcal{E})$  with radius  $r = (\prod_{i=1}^{n-1} b_i)^{2/(n-1)}$ .

Property (i) of the lemma is preserved:

$$N_{T_2(K_1)}(T_2(x_1)) = N_{K_1}(x_1) = e_n.$$

Indeed, by Lemma 3

$$N_{T_2(K_1)}(T_2(x_1)) = \frac{(T_2^{-1})^t(N_{K_1}(x_1))}{\|(T_2^{-1})^t(N_{K_1}(x_1))\|}.$$

and thus for all  $1 \leq i \leq n-1$

$$\langle (T_2^{-1})^t(N_{K_1}(x_1)), e_i \rangle = \langle N_{K_1}(x_1), T_2^{-1}(e_i) \rangle = \left\langle e_n, a_1 \left( \prod_{i=1}^{n-1} b_i \right)^{-\frac{2}{n-1}} e_i \right\rangle = 0.$$

(iii) It is enough to apply a multiple  $\alpha I$  of the identity.

(iv) We apply the map  $T_3$  with

$$T_3(\xi) = (\lambda \xi_1, \dots, \lambda \xi_{n-1}, \lambda^{-n+1} \xi_n)$$

where

$$\lambda = (\alpha \langle x_0, e_n \rangle)^{1/(n-1)}$$

Properties (i) and (ii) of the lemma are preserved and, as  $\det(T_3) = 1$ , Property (iii) as well. Finally, we let  $T(K) = T_3(\alpha T_2(K_1))$ .  $\square$

**Lemma 7.** *Let  $K$  be a convex body in  $\mathbb{R}^n$  such that  $\partial K$  is twice differentiable in the generalized sense at  $x$ . Suppose that  $\|x\| = 1$ ,  $N_K(x) = x$ , and the indicatrix of Dupin at  $x$  is a Euclidean sphere with radius  $r$ . Then  $x \in \partial K^\circ$  and for all  $0 < \epsilon < \min\{r, \frac{1}{r}\}$  there is  $\Delta > 0$  such that*

$$B_2^n \left( x - \left( \frac{1}{r} - \epsilon \right) N_{K^\circ}(x), \frac{1}{r} - \epsilon \right) \cap H^-(x - \Delta N_{K^\circ}(x), N_{K^\circ}(x))$$

$$\begin{aligned} &\subseteq K^\circ \cap H^-(x - \Delta N_{K^\circ}(x), N_{K^\circ}(x)) \\ &\subseteq B_2^n \left( x - \left( \frac{1}{r} + \epsilon \right) N_{K^\circ}(x), \frac{1}{r} + \epsilon \right) \cap H^-(x - \Delta N_{K^\circ}(x), N_{K^\circ}(x)) \end{aligned}$$

**Proof.** Without loss of generality we can assume that  $x = N_{K(x)} = e_n$ . Clearly then  $x \in \partial K^\circ$  and  $N_{K^\circ(x)} = x$ . Let  $0 < \epsilon < \min\{r, \frac{1}{r}\}$ . By Lemma 5, there exists  $\Delta_1$  such that for all  $\Delta \leq \Delta_1$

$$\begin{aligned} &B_2^n((1 - (r - \epsilon))e_n, r - \epsilon) \cap H^-((1 - \Delta)e_n, e_n) \\ &\subseteq K \cap H^-((1 - \Delta)e_n, e_n) \subseteq B_2^n((1 - (r + \epsilon))e_n, r + \epsilon) \cap H^-((1 - \Delta)e_n, e_n). \end{aligned}$$

We construct two new convex bodies.

$$K_1 = \text{co} \left[ K \cap H^+((1 - \Delta_1)e_n, e_n), B_2^n((1 - (r - \epsilon))e_n, r - \epsilon) \right]$$

and

$$K_2 = \text{co} \left[ K \cap H^+((1 - \Delta_1)e_n, e_n), B_2^n((1 - (r + \epsilon))e_n, r + \epsilon) \right].$$

Then  $K_1 \subseteq K \subseteq K_2$  and there is  $\Delta_2 \leq \Delta_1$  such that for all  $\Delta \leq \Delta_2$

$$K_1 \cap H^-((1 - \Delta)e_n, e_n) = B_2^n((1 - (r - \epsilon))e_n, r - \epsilon) \cap H^-((1 - \Delta)e_n, e_n)$$

and

$$K_2 \cap H^-((1 - \Delta)e_n, e_n) = B_2^n((1 - (r + \epsilon))e_n, r - \epsilon) \cap H^-((1 - \Delta)e_n, e_n).$$

We now compute  $K_1^0$  and  $K_2^0$  in a neighborhood of  $x = e_n$ . We show the computations for  $K_1$ .  $K_2$  is done similarly.

Let  $\Delta \leq \Delta_2$  and  $\eta$  be the normal of  $y \in \partial K_1 \cap H^-((1 - \Delta)e_n, e_n)$ . Then

$$\langle y, \eta \rangle = r - \epsilon + (1 - (r - \epsilon)) \langle x, \eta \rangle$$

Therefore,

$$\left\langle y, \frac{\eta}{r - \epsilon + (1 - (r - \epsilon)) \langle x, \eta \rangle} \right\rangle = 1$$

and hence

$$\frac{\eta}{r - \epsilon + (1 - (r - \epsilon)) \langle x, \eta \rangle} \in \partial K_1^\circ.$$

For  $\Delta_{K_1^\circ} \leq \Delta_2$  sufficiently small, we consider now a cap of  $K_1^\circ$  and its base

$$K_1^\circ \cap H((1 - \Delta_{K_1^\circ})e_n, e_n)$$

We compute the distance  $\rho$  of  $\frac{\eta}{r - \epsilon + (1 - (r - \epsilon)) \langle x, \eta \rangle} \in \partial K_1^\circ \cap H((1 - \Delta_{K_1^\circ})e_n, e_n)$  from the center of the base  $(1 - \Delta_{K_1^\circ})e_n$ . Clearly, by Pythagoras

$$\rho = \sqrt{\frac{1}{(r - \epsilon + (1 - (r - \epsilon)) \langle x, \eta \rangle)^2} - (1 - \Delta_{K_1^\circ})^2}$$

Moreover,

$$1 - \Delta_{K_1^\circ} = \frac{\langle x, \eta \rangle}{r - \epsilon + [1 - (r - \epsilon)] \langle x, \eta \rangle}$$

Hence

$$\begin{aligned} (1 - \Delta_{K_1^\circ}) \left( r - \epsilon + [1 - (r - \epsilon)] \langle x, \eta \rangle \right) &= \langle x, \eta \rangle \\ \langle x, \eta \rangle \left( (1 - \Delta_{K_1^\circ}) [1 - (r - \epsilon)] - 1 \right) &= -(r - \epsilon)(1 - \Delta_{K_1^\circ}) \\ \langle x, \eta \rangle &= \frac{(r - \epsilon)(1 - \Delta_{K_1^\circ})}{r - \epsilon + [1 - (r - \epsilon)] \Delta_{K_1^\circ}} \end{aligned}$$

Therefore

$$\rho = \sqrt{\frac{(r - \epsilon + [1 - (r - \epsilon)] \Delta_{K_1^\circ})^2}{(r - \epsilon)^2} - (1 - \Delta_{K_1^\circ})^2}$$

or

$$\rho = \sqrt{\frac{2}{r - \epsilon} \Delta_{K_1^\circ} + \frac{[1 - 2(r - \epsilon)]}{(r - \epsilon)^2} \Delta_{K_1^\circ}^2}$$

We compare this radius with the corresponding radius of the ball  $B_2^n((1 - \frac{1}{r-2\epsilon})e_n, \frac{1}{r-2\epsilon})$ . The corresponding radius is

$$\sqrt{\frac{2\Delta_{K_1^\circ}}{r - 2\epsilon} - \Delta_{K_1^\circ}^2}$$

Therefore,

$$\rho \leq \sqrt{\frac{2\Delta_{K_1^\circ}}{r - 2\epsilon} - \Delta_{K_1^\circ}^2}$$

provided that

$$\Delta_{K_1^\circ} \leq \min \left\{ \frac{2\epsilon}{(r - 2\epsilon)(r - \epsilon)}, \frac{2\epsilon}{(r - 2\epsilon)(2 + r - \epsilon)} \right\}.$$

As  $K_1 \subseteq K$  and consequently  $K^\circ \subseteq K_1^\circ$  we get

$$K^\circ \cap H^-((1 - \Delta_{K_1^\circ})e_n, e_n) \subseteq B_2^n \left( \left(1 - \frac{1}{r - 2\epsilon}\right) e_n, \frac{1}{r - 2\epsilon} \right) \cap H^-((1 - \Delta_{K_1^\circ})e_n, e_n).$$

Similarly, using  $K \subseteq K_2$ , one shows (with a new  $\Delta_{K_1^\circ}$  small enough if needed) that

$$B_2^n \left( \left(1 - \frac{1}{r + 2\epsilon}\right) e_n, \frac{1}{r + 2\epsilon} \right) \cap H^-((1 - \Delta_{K_1^\circ})e_n, e_n) \subseteq K^\circ \cap H^-((1 - \Delta_{K_1^\circ})e_n, e_n).$$

□

Lemma 7 implies that if  $x \in \partial K$  is a twice differentiable point (in the generalized sense), then the point  $y \in \partial K^\circ$  with  $\langle x, y \rangle = 1$ , is also a twice differentiable point. Compare also [6].



**Corollary 8.** *Let  $K$  be a convex body in  $\mathbb{R}^n$  with  $0$  as an interior point. Assume that  $\partial K$  is twice differentiable in the generalized sense at  $x$  and the indicatrix of Dupin at  $x$  is an ellipsoid (and not a cylinder with an ellipsoid as its base). Then  $\partial K^\circ$  is twice differentiable at the unique point  $\xi$  with  $\langle \xi, x \rangle = 1$ .*

**Proof.** By Lemma 6 we may assume that the indicatrix is a Euclidean ball and that  $N_K(x) = x$ . By Lemma 7 the statement follows.  $\square$

The next lemma is also well known (see e.g. [17]).

**Lemma 9.** *Let  $K$  be a convex body in  $\mathbb{R}^n$  and suppose that the indicatrix of Dupin at  $x \in \partial K$  exists and is a Euclidean ball of radius  $r > 0$ . Let  $C(r, \Delta)$  be the cap at  $x$  of height  $\Delta$ . Then*

$$|C(r, \Delta)| = g\left(\frac{\Delta}{r}\right)^{\frac{n+1}{2}} \frac{1}{n+1} 2^{\frac{n+1}{2}} \text{vol}_{n-1}(B_2^{n-1}) \Delta^{\frac{n+1}{2}} r^{\frac{n-1}{2}}$$

where  $\lim_{t \rightarrow 0} g(t) = 1$ .

**Remark.** The conclusions of Lemma 9 also hold if instead of the existence of the indicatrix, we assume the following:

Let  $x \in \partial K$  and suppose that there is  $r > 0$  such that for all  $\epsilon > 0$  there is a  $\Delta_\epsilon$  such that for all  $\Delta$  with  $0 < \Delta \leq \Delta_\epsilon$

$$\begin{aligned} & B_2^n(x - (r - \epsilon)N_K(x), r - \epsilon) \cap H^-(x - \Delta N_K(x), N_K(x)) \\ & \subseteq K \cap H^-(x - \Delta N_K(x), N_K(x)) \\ & \subseteq B_2^n(x - (r + \epsilon)N_K(x), r + \epsilon) \cap H^-(x - \Delta N_K(x), N_K(x)). \end{aligned} \quad (6)$$

The next lemma is from [21]. There it was assumed that the indicatrix of Dupin at  $x \in \partial K$  exists and is a Euclidean ball of radius  $r > 0$ . However, what was actually used in the proof, were the assumptions (6) of the above Remark.

**Lemma 10.** [21] *Let  $K$  be a convex body in  $\mathbb{R}^n$ . Let  $x \in \partial K$  and suppose that there is  $r > 0$  such that for all  $\epsilon > 0$  there is a  $\Delta_\epsilon$  such that for all  $\Delta$  with  $0 < \Delta \leq \Delta_\epsilon$ , (6) holds. Then, if  $\Delta_\epsilon$  is small enough, we have for  $0 < \Delta < \Delta_\epsilon$*

$$\begin{aligned} & \frac{2^{\frac{n+1}{2}}}{n(n+1)} |B_2^{n-1}| \Delta^{\frac{n+1}{2}} r^{\frac{n-1}{2}} \\ & \quad \times \left\{ (n+1) \left(1 - \frac{\epsilon}{r}\right)^{\frac{n-1}{2}} (1 - c\epsilon) - n \left(1 + \frac{\epsilon}{r}\right)^{\frac{n-1}{2}} (1 + c\epsilon) h\left(\frac{\Delta}{r + \epsilon}\right)^{n+1} \right\} \\ & \leq \left| K^x(\Delta) \setminus K \right| \\ & \leq \frac{2^{\frac{n+1}{2}}}{n(n+1)} |B_2^{n-1}| \Delta^{\frac{n+1}{2}} r^{\frac{n-1}{2}} \end{aligned}$$

$$\times \left\{ (n+1) \left(1 + \frac{\epsilon}{r}\right)^{\frac{n-1}{2}} (1 + c \epsilon) - n \left(1 - \frac{\epsilon}{r}\right)^{\frac{n-1}{2}} (1 - c \epsilon) h \left( \frac{\Delta}{r + \epsilon} \right)^{n+1} \right\}.$$

where  $c$  is a constant and  $\lim_{t \rightarrow 0} h(t) = 1$ .

### Proof of Theorem 1.

By assumption there is a point  $x \in \partial K$  at which  $\partial K$  is twice differentiable in the generalized sense. By Lemma 6 we may assume that  $\|x\| = 1$  and  $x = N_K(x)$ . Moreover, all principal curvature radii at  $x$  are equal to  $r$ . By Lemma 7,  $x \in \partial K^\circ$ ,  $K^\circ$  is twice differentiable at  $x$  and all principal curvature radii are equal to  $\frac{1}{r}$ .

The dual body to  $K_x(\Delta)$  is  $K^x(\frac{\Delta}{1-\Delta})$ . By Lemma 9,

$$|K_x(\Delta)| \leq |K| - g \left( \frac{\Delta}{r} \right)^{\frac{n+1}{2}} \frac{2^{\frac{n+1}{2}}}{n+1} \text{vol}_{n-1}(B_2^{n-1}) \Delta^{\frac{n+1}{2}} r^{-\frac{n-1}{2}}.$$

By Lemma 10

$$\begin{aligned} |K_x(\Delta)^\circ| &= \left| K^x \left( \frac{\Delta}{1-\Delta} \right) \right| \leq |K^\circ| + \frac{2^{\frac{n+1}{2}}}{n(n+1)} |B_2^{n-1}| \Delta^{\frac{n+1}{2}} r^{-\frac{n-1}{2}} \\ &\times \left\{ (n+1) (1 + r\epsilon)^{\frac{n-1}{2}} (1 + c \epsilon) - n (1 - r\epsilon)^{\frac{n-1}{2}} (1 - c \epsilon) h \left( \frac{\Delta}{\frac{1}{r} + \epsilon} \right)^{n+1} \right\}. \end{aligned}$$

It follows that

$$\begin{aligned} |K_x(\Delta)| |K_x(\Delta)^\circ| &\leq |K| |K^\circ| + |K| \frac{2^{\frac{n+1}{2}}}{n(n+1)} |B_2^{n-1}| \Delta^{\frac{n+1}{2}} r^{-\frac{n-1}{2}} \\ &\times \left\{ (n+1) (1 + r\epsilon)^{\frac{n-1}{2}} (1 + c \epsilon) - n (1 - r\epsilon)^{\frac{n-1}{2}} (1 - c \epsilon) h \left( \frac{\Delta}{\frac{1}{r} + \epsilon} \right)^{n+1} \right\} \\ &- |K^\circ| g \left( \frac{\Delta}{r} \right)^{\frac{n+1}{2}} \frac{1}{n+1} 2^{\frac{n+1}{2}} \text{vol}_{n-1}(B_2^{n-1}) \Delta^{\frac{n+1}{2}} (r - \epsilon)^{\frac{n-1}{2}}. \end{aligned}$$

Therefore we have

$$|K_x(\Delta)| |K_x(\Delta)^\circ| < |K| |K^\circ|,$$

provided that

$$\begin{aligned} &|K| \left\{ (n+1) (1 + r\epsilon)^{\frac{n-1}{2}} (1 + c \epsilon) - n (1 - r\epsilon)^{\frac{n-1}{2}} (1 - c \epsilon) h \left( \frac{\Delta}{\frac{1}{r} + \epsilon} \right)^{n+1} \right\} \\ &< n |K^\circ| g \left( \frac{\Delta}{r} \right)^{\frac{n+1}{2}} r^{n-1} \left( 1 - \frac{\epsilon}{r} \right)^{\frac{n-1}{2}}. \end{aligned} \tag{7}$$

Now we interchange the roles of  $K$  and  $K^\circ$ . We cut off a cap from  $K^\circ$  and apply the remark following Lemma 9. Then the inequality analogous to (7) will be

$$|K^\circ| \left\{ (n+1) \left(1 + \frac{\epsilon}{r}\right)^{\frac{n-1}{2}} (1 + c \epsilon) - n \left(1 - \frac{\epsilon}{r}\right)^{\frac{n-1}{2}} (1 - c \epsilon) h \left( \frac{\Delta}{r + \epsilon} \right)^{n+1} \right\}$$

$$< n|K|g(r\Delta)^{\frac{n+1}{2}}r^{-(n-1)}(1-r\epsilon)^{\frac{n-1}{2}}. \quad (8)$$

Thus the theorem is proved provided that one of the inequalities (7) or (8) holds. Suppose both inequalities do not hold. Then

$$\begin{aligned} & |K| \left\{ (n+1)(1+r\epsilon)^{\frac{n-1}{2}}(1+c\epsilon) - n(1-r\epsilon)^{\frac{n-1}{2}}(1-c\epsilon) h\left(\frac{\Delta}{\frac{1}{r}+\epsilon}\right)^{n+1} \right\} \\ & \geq n|K^\circ| g\left(\frac{\Delta}{r}\right)^{\frac{n+1}{2}} r^{n-1} \left(1-\frac{\epsilon}{r}\right)^{\frac{n-1}{2}} \\ & \geq n^2|K| \frac{g\left(\frac{\Delta}{r}\right)^{\frac{n+1}{2}} \left(1-\frac{\epsilon}{r}-r\epsilon+\epsilon^2\right)^{\frac{n-1}{2}} g(r\Delta)^{\frac{n+1}{2}}}{\left\{ (n+1)(1+\frac{\epsilon}{r})^{\frac{n-1}{2}}(1+c\epsilon) - n\left(1-\frac{\epsilon}{r}\right)^{\frac{n-1}{2}}(1-c\epsilon) h\left(\frac{\Delta}{r+\epsilon}\right)^{n+1} \right\}}. \end{aligned}$$

We can choose  $\epsilon$  so small that

$$(n+1)(1+r\epsilon)^{\frac{n-1}{2}}(1+c\epsilon) - n(1-r\epsilon)^{\frac{n-1}{2}}(1-c\epsilon) h\left(\frac{\Delta}{\frac{1}{r}+\epsilon}\right)^{n+1} \leq 2$$

and

$$(n+1)\left(1+\frac{\epsilon}{r}\right)^{\frac{n-1}{2}}(1+c\epsilon) - n\left(1-\frac{\epsilon}{r}\right)^{\frac{n-1}{2}}(1-c\epsilon) h\left(\frac{\Delta}{r+\epsilon}\right)^{n+1} \leq 2.$$

Moreover, we can choose  $\epsilon$  so small that

$$\left(1-\frac{\epsilon}{r}-r\epsilon+\epsilon^2\right)^{\frac{n-1}{2}} \geq \frac{1}{2}$$

Therefore

$$4 \geq n^2 g\left(\frac{\Delta}{r}\right)^{\frac{n+1}{2}} g(r\Delta)^{\frac{n+1}{2}}$$

Since  $\lim_{t \rightarrow 0} g(t) = 1$ , this gives a contradiction.

The extension of the proof needed in order to prove the symmetric case is obvious.  $\square$

## References

- [1] A.D. Alexandrov, *Almost everywhere existence of the second differential of a convex function and some properties of convex surfaces connected with it*, Uchen. Zap. Leningrad Gos. Univ. Mat. Ser. **6** (1939), 3-35.
- [2] W. Blaschke, *Über affine Geometrie VII. Neue Extremeigenschaften von Ellipse und Ellipsoid*, Leipz. Ber., **69** (1917), 306-318.
- [3] J. Bourgain and V. D. Milman, *New volume ratio properties for convex symmetric bodies in  $\mathbb{R}^n$* , Invent. Math. **88** (1987), 319-340.
- [4] H. Busemann and W. Feller, *Krümmungseigenschaften konvexer Flächen*, Acta Math. **66** (1936), 1-47.

- [5] Y. Gordon, M. Meyer and S. Reisner, *Zonoids with minimal volume product—a new proof*, Proc. Amer. Math. Soc. **104** (1988), 273-276.
- [6] D. HUG, *Curvature Relations and Affine Surface Area for a General Convex Body and its Polar*, Results in Mathematics **V. 29** (1996), 233-248.
- [7] G. Kuperberg, *From the Mahler Conjecture to Gauß Linking Integrals*, Geometric And Functional Analysis **18** (2008), 870-892.
- [8] K. Mahler, *Ein Minimalproblem für konvexe Polygone*, Mathematica (Zutphen) **B 7** (1939), 118-127.
- [9] K. Mahler, *Ein Übertragungsprinzip für konvexe Körper*, Časopis Pěst. Mat. Fys. **68** (1939), 93-102.
- [10] M. Meyer, *Une caractérisation volumique de certains espaces normés*, Israel J. Math. **55** (1986), 317-326.
- [11] M. Meyer and A. Pajor, *On the Blaschke-Santaló inequality*, Arch. Math. **55** (1990), 82-93.
- [12] F. Nazarov, *The Hörmander proof of the Bourgain-Milman theorem*, preprint, 2009.
- [13] S. Reisner, *Zonoids with minimal volume-product*, Math. Z. **192** (1986), 339-346.
- [14] S. Reisner, *Minimal volume product in Banach spaces with a 1-unconditional basis*, J. London Math. Soc. **36** (1987), 126-136.
- [15] J. Saint-Raymond, *Sur le volume des corps convexes symétriques*, Séminaire d'Initiation à l'Analyse, 1980-1981, Université PARIS VI, Paris 1981.
- [16] L. A. Santaló, *Un invariante afin para los cuerpos convexos del espacio de  $n$  dimensiones*, Portugal. Math. **8** (1949), 155-161.
- [17] C. Schütt and E. Werner, *The convex floating body*, Math. Scand. **66** (1990), 275-290.
- [18] C. Schütt and E. Werner, *Random polytopes of points chosen from the boundary of a convex body* GAFA Seminar Notes, Lecture Notes in Math., **1807**, Springer-Verlag, (2002), 241-422.
- [19] C. Schütt and E. Werner, *Surface bodies and  $p$ -affine surface area*, Advances in Mathematics **187** (2004), 98-145.
- [20] A. Stancu, *Two volume product inequalities and their applications*, Canad. Math. Bull. **52** (2009), 464-472.
- [21] E. Werner, *Illumination bodies and affine surface area*, Studia Mathematica **110** (1994), 257-269.

Shlomo Reisner  
Department of Mathematics  
University of Haifa  
Haifa, 31905 Israel  
`reisner@math.haifa.ac.il`

Carsten Schütt  
Mathematisches Seminar  
Christian-Albrechts Universität  
D-24098 Kiel, Ludewig-Meyn Strasse 4, Germany  
`schuett@math.uni-kiel.de`

Elisabeth Werner  
Department of Mathematics  
Case Western Reserve University  
Cleveland, Ohio 44106, U. S. A.  
`elisabeth.werner@case.edu`

Université de Lille 1  
UFR de Mathématique  
59655 Villeneuve d'Ascq, France